Problem 1. Use the definition of the derivative as the limit of a difference qoutient to compute the derivative of the functions below

(i) $f(x) = \sqrt{x}$

Solution. (i) We have

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Problem 2. Find the derivative of $f(x)$

(iii) $f(x) = x^{42}$

(v) $f(x) = \dfrac{1}{\sqrt[5]{x^4}}$

(ix) $f(x) = x^4 - 2x^3 + 5x - 7$

Solution. The key is the power rule

$$f(x) = x^n, \quad f'(x) = nx^{n-1}.$$

(iii) $f(x) = x^{42}$. We use the power rule.

$$f'(x) = 42x^{42-1} = 42x^{41}$$

(vi) $f(x) = \dfrac{1}{\sqrt[5]{x^4}} = x^{-\frac{4}{5}}$. We use the power rule.

$$f'(x) = -\frac{4}{5}x^{-\frac{4}{5}-1} = -\frac{4}{5}x^{-\frac{9}{5}} = -\frac{4}{5x\sqrt[5]{x^4}}$$

(ix) $f(x) = x^4 - 2x^3 + 5x - 7$. We use the power rule and the sum/difference rule.

$$f'(x) = 4x^{4-1} - 3 \cdot 2x^{3-1} + 5x^{1-1} - 0 = 4x^3 - 6x^2 + 5$$

Problem 3. Find the derivative of $f(x)$

(i) $f(x) = \sin x + \cos x$

(iii) $f(x) = -5e^x$

(v) $f(x) = 2e^x - 3\cos x + 5\sin x - 6e^7$

Solution. We use the sum/difference rule, the constant multiple rule and

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}e^x = e^x$$

(i) $f(x) = \sin x + \cos x$.

$$f'(x) = \cos x - \sin x$$

(iii) $f(x) = -5e^x$

$$f'(x) = -5e^x$$

(v) $f(x) = 2e^x - 3\cos x + 5\sin x - 6e^7$

$$f'(x) = 2e^x - 3(-\sin x) + 5\cos x - 0 = 2e^x + 3\sin x + 5\cos x$$

Problem 4.

(i) Standing on the roof of a 72 feet tall building, a golfer takes a whack at a ball sending it over the edge. The initial velocity of the ball is 24 feet per second and the acceleration due to gravity is -32 feet per second per second.

(a) Determine the position function $s(t)$, that is the height of the ball above the ground as a function of time in seconds.

(b) Determine the velocity function.

(c) What is the velocity of the ball when it hits the ground?

Solution. (i) The position function of a free falling object is given by

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0.$$

We have acceleration due to gravity $g = -32\frac{\text{ft}}{\text{sec}^2}$, initial velocity $v_0 = 24\frac{\text{ft}}{\text{sec}}$ and initial position $s_0 = 72\text{ft}$. This means that the position function is given by

$$s(t) = -16t^2 + 24t + 72.$$

(ii) The velocity function is the first derivative of the position function, so

$$v(t) = s'(t) = -32t + 24.$$

(iii) The ball hits the ground when position function is zero. That is

$$\begin{aligned} s(t) &= -16t^2 + 24t + 72 = 0, \\ &= -8(2t^2 - 3t - 9) = 0, \\ &= (2t+3)(t-3) = 0. \end{aligned}$$

This means that the ball hits the ground at time $t = 3$ sec. At that time, the velocity is

$$v(3) = -32(3) + 24 = -72 \frac{\text{ft}}{\text{sec}}.$$

Problem 5. Use the quotient rule to prove the formulas for the derivatives of the trigonometric functions

(iii) $\frac{d}{dx} \csc x = -\csc x \cot x$

Solution. We use the quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$\begin{aligned} \frac{d}{dx} \csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\ &= \frac{\sin x \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(\sin x)}{\sin^2 x} \\ &= \frac{\sin x \cdot 0 - \cos x}{\sin^2 x} \\ &= -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} \\ &= -\csc x \cot x \end{aligned}$$

Problem 6. Find the derivative of $f(x)$.

(ii) $f(x) = \sin x \cos x$

(v) $f(x) = x \csc x$

(viii) $f(x) = \dfrac{\sin x - \cos x}{e^x + x}$

Solution. We'll use the product rule

$$\frac{d}{dx}(f(x)g(x)) = g(x)f'(x) + f(x)g'(x)$$

and the quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

(ii) $f(x) = \sin x \cos x$

$$\begin{aligned} f'(x) &= \sin x \cos x \\ &= \cos x \cdot \frac{d}{dx}\sin x + \sin x \cdot \frac{d}{dx}\cos x \\ &= \cos x \cos x + \sin x(-\sin x) \\ &= \cos^2 x - \sin^2 x \\ &= \cos(2x) \end{aligned}$$

(v) $f(x) = x \csc x$

$$\begin{aligned} f'(x) &= \csc x \frac{d}{dx}(x) + x\frac{d}{dx}(\csc x) \\ &= \csc x \cdot 1 + x(-\csc x \cot x) \\ &= \csc x - x \csc x \cot x \\ &= \csc x(1 - x\cot x) \end{aligned}$$

(viii) $f(x) = \dfrac{\sin x - \cos x}{e^x + x}$

$$\begin{aligned} f'(x) &= \frac{(e^x + x)\frac{d}{dx}(\sin x - \cos x) - (\sin x - \cos x)\frac{d}{dx}(e^x + x)}{(e^x + x)^2} \\ &= \frac{(e^x + x)(\cos x + \sin x) - (\sin x - \cos x)(e^x + 1)}{(e^x + x)^2} \\ &= \frac{(2e^x + x + 1)\cos x + (x - 1)\sin x}{(e^x + x)^2} \end{aligned}$$

Problem 7. Find the derivative of $f(x)$. 6

(i) $f(x) = \sin(2x)$

(iv) $f(x) = e^{\sqrt{x}}$

(vii) $f(x) = \ln\left(\dfrac{\sqrt{2x^2 - 3x + 2}}{(x^2 + 1)^2}\right)$

Solution. (i) We have

$$f'(x) = \cos(2x) \cdot \frac{d}{dx}(2x) = \cos(2x) \cdot 2 = 2\cos(2x).$$

(iv) We have

$$f'(x) = e^{\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x}) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}.$$

(vii) We have

$$\begin{aligned}
f'(x) &= \frac{d}{dx} \ln\left(\frac{\sqrt{2x^2 - 3x + 2}}{(x^2+1)^2}\right) \\
&= \frac{d}{dx}\left(\ln\sqrt{2x^2 - 3x + 2} - \ln(x^2+1)^2\right) \\
&= \frac{d}{dx}\left(\ln(2x^2 - 3x + 2)^{\frac{1}{2}} - 2\ln(x^2+1)\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}\ln(2x^2 - 3x + 2) - 2\ln(x^2+1)\right) \\
&= \frac{1}{2}\frac{d}{dx}\ln(2x^2 - 3x + 2) - 2\frac{d}{dx}\ln(x^2+1) \\
&= \frac{1}{2}\frac{1}{2x^2 - 3x + 2} \cdot \frac{d}{dx}(2x^2 - 3x + 2) - 2\frac{1}{x^2+1} \cdot \frac{d}{dx}(x^2+1) \\
&= \frac{1}{2}\frac{1}{2x^2 - 3x + 2}(4x - 3) - 2\frac{1}{x^2+1}(2x) \\
&= \frac{4x - 3}{4x^2 - 6x + 4} - \frac{4x}{x^2+1}
\end{aligned}$$

I'm fine with that, but if it pleases, we could combine the fractions to

$$\begin{aligned}
f'(x) &= \frac{(4x-3)(x^2+1)}{(4x^2 - 6x + 4)(x^2+1)} - \frac{4x(4x^2 - 6x + 4)}{(x^2+1)(4x^2 - 6x + 4)} \\
&= \frac{-3(4x^3 - 7x^2 + 4x + 1)}{2(x^4 - 3x^3 + 4x^2 - 3x + 2)}
\end{aligned}$$