Problem 1. Find the limit, given it exists.

(ii) $\lim_{x\to 1} \frac{x^3 - x^2 + x - 1}{x+2}$

(iii) $\lim_{x\to 0} \frac{|x|}{x}$

(vi) $\lim_{x\to -1} f(x)$,
where $f(x) = \begin{cases} x+1, & \text{if } x \neq -1 \\ 42, & \text{if } x = -1 \end{cases}$

(vii) $\lim_{x\to 3} \frac{2x^2 - 5x - 3}{x-3}$

(ix) $\lim_{x\to 0} \frac{\sin(5x)}{3x}$

(x) $\lim_{x\to 0} \frac{\sin(2x)e^x}{3x\cos x}$

Solution. (ii) The function is defined and continuous at $x = 1$, so the limit is the same as the function value.

$$\lim_{x\to 1} \frac{x^3 - x^2 + x - 1}{x+2} = \frac{1^3 - 1^2 + 1 - 1}{1+2} = \frac{0}{3} = 0.$$

(iii) When $x \to 0$ from the left,

$$\frac{|x|}{x} = \frac{-x}{x} = -1.$$

When $x \to 0$ from the right,

$$\frac{|x|}{x} = \frac{x}{x} = 1.$$

So the limit does not exist.

(vi) $f(x) = x + 1$ for all x in an interval around -1, so by Thm. 2.7

$$\lim_{x\to -1} f(x) = \lim_{x\to -1}(x+1) = 0.$$

(vii) Direct substitution yields $\frac{0}{0}$. By Thm. 2.7

$$\lim_{x\to 3} \frac{2x^2 - 5x - 3}{x-3} = \lim_{x\to 3} \frac{(x-3)(2x+1)}{x-3} = \lim_{x\to 3}(2x+1) = 2\cdot 3 + 1 = 7.$$

(ix) By Thm. 2.9, we have $\lim_{t\to 0} \frac{\sin t}{t} = 1$. So

$$\lim_{x\to 0} \frac{\sin(5x)}{3x} = \lim_{x\to 0} \frac{5\sin(5x)}{3\cdot 5x} = \frac{5}{3} \lim_{x\to 0} \frac{\sin(5x)}{5x} = \frac{5}{3} \lim_{5x\to 0} \frac{\sin(5x)}{5x}$$
$$= \frac{5}{3} \lim_{t\to 0} \frac{\sin t}{t} = \frac{5}{3}.$$

(x) We have

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(2x)e^x}{3x\cos x} &= \lim_{x\to 0}\frac{2}{3}\cdot\frac{\sin(2x)}{2x}\cdot\frac{e^x}{\cos x} = \frac{2}{3}\left(\lim_{x\to 0}\frac{\sin(2x)}{2x}\right)\left(\lim_{x\to 0}\frac{e^x}{\cos x}\right)\\ &= \frac{2}{3}\cdot 1\cdot 1 = \frac{2}{3}.\end{aligned}$$

Problem 2. Find the discontinuities (if any) for each of the functions below. Determine whether they are removable, and, in that case, (re)define the function value to make it continuous at that point.

(ii) $f(x) = \dfrac{x-4}{\sqrt{x-3}-1}$ (iii) $f(x) = \dfrac{2}{x^2-1}$ (iv) $f(x) = \dfrac{x+1}{|x-1|}$

Solution. (ii) $f(x)$ is a combination of elementary functions, so it is continuous everywhere in its domain. x can't be smaller than 3 since that would cause a negative under the square root. $x = 4$ yields a zero divisor and is the only point ≥ 3 where f is not defined, hence the only discontinuity. We have

$$\begin{aligned}\lim_{x\to 4}\frac{x-4}{\sqrt{x-3}-1} &= \lim_{x\to 4}\frac{x-4}{\sqrt{x-3}-1}\cdot\frac{\sqrt{x-3}+1}{\sqrt{x-3}+1}\\ &= \lim_{x\to 4}\frac{(x-4)(\sqrt{x-3}+1)}{(\sqrt{x-3}-1)(\sqrt{x-3}+1)}\\ &= \lim_{x\to 4}\frac{(x-4)(\sqrt{x-3}+1)}{x-3-1}\\ &= \lim_{x\to 4}\frac{\cancel{(x-4)}(\sqrt{x-3}+1)}{\cancel{x-4}}\\ &= \lim_{x\to 4}(\sqrt{x-3}+1)\\ &= \sqrt{4-3}+1 = 2.\end{aligned}$$

By setting $f(4) = 2$, f is made continuous at $x = 4$, so this is a removable discontinuity.

(iii) $f(x)$ is a rational function, so it is continuous everywhere in its domain. $x = \pm 1$ yield a zero divisor. These are the only discontinuities.

$$\lim_{x\to\pm 1} f(x) = \infty.$$

Since the limit doesn't exist, the discontinuities are non-removable.

(iv) $x = 1$ yields a zero divisor. This is the only discontinuity. Since the numerator is $2 \neq 0$ at $x = 1$, there is a vertical asymptote at $x = 1$, so this is a nonremovable discontinuity.

Problem 3. For each of the functions below, determine the value k for which the function is everywhere continuous.

(i) $f(x) = \begin{cases} \frac{x^2+4x+3}{x+1}, & \text{if } x \neq -1 \\ k, & \text{if } x = -1 \end{cases}$

Solution. (i) $\frac{x^2+4x+3}{x+1}$ only has a discontinuity at $x = -1$. The limit at this point is

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x^2+4x+3}{x+1} = \lim_{x\to -1} \frac{(x+1)(x+3)}{x+1} = 2,$$

so if we put $k = 2$, we have

$$\lim_{x\to -1} f(x) = f(-1),$$

which means that $f(x)$ is continuous at $x = -1$, hence everywhere.

Problem 4. Find all the vertical asymptotes (if any) of the functions below.

(i) $f(x) = \dfrac{1}{x^2-2x+1}$ (iv) $f(x) = \dfrac{x+1}{x^2-1}$

Solution. (i) The numerator is nonzero, so there are vertical asymptotes where the denominator is zero. That is

$$x^2 - 2x + 1 = (x-1)^2 = 0,$$

so $x = 1$.

(iv) The denominator is zero at

$$x^2 - 1 = (x+1)(x-1) = 0, \quad x = \pm 1.$$

At $x = -1$, the numerator is zero also. This is a removable discontinuity, so not a vertical asymptote. The only vertical asymptote is $x = 1$.

Problem 5. Find the one-sided limit, given it exists (indicate unboundedness by $\pm\infty$).

(ii) $\displaystyle\lim_{x\to 1^-} \frac{x-1}{|x-1|}$, $\displaystyle\lim_{x\to 1^+} \frac{x-1}{|x-1|}$ (iv) $\displaystyle\lim_{x\to 2^-} \frac{2x-4}{|x-2|}$, $\displaystyle\lim_{x\to 2^+} \frac{2x-4}{|x-2|}$

Solution. (ii) When x approaches 1 from the left, $x - 1 < 0$, so

$$\lim_{x\to 1^-} \frac{x-1}{|x-1|} = \lim_{x\to 1^-} \frac{x-1}{-(x-1)} = -1.$$

When x approaches 1 from the right, $x-1>0$, so

$$\lim_{x\to 1^+} \frac{x-1}{|x-1|} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1.$$

(iv) When x approaches 2 from the left, $x-2<0$, so

$$\begin{aligned}\lim_{x\to 2^-} \frac{2x-4}{|x-2|} &= \lim_{x\to 2^-} \frac{2(x-2)}{-(x-2)} \\ &= \lim_{x\to 2^-} \frac{2}{-1} = -2.\end{aligned}$$

When x approaches 2 from the right, $x-2>0$, so

$$\begin{aligned}\lim_{x\to 2^+} \frac{2x-4}{|x-2|} &= \lim_{x\to 2^+} \frac{2(x-2)}{x-2} \\ &= \lim_{x\to 2^+} \frac{2}{1} = 2.\end{aligned}$$